Géométrie semi-riemannienne

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Chapitre 1

Differentiable manifolds

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1.1. Differentiable manifolds

Let S be a topological connected Hausdorff and paracompact space. A *chart* is a homeomorphism ξ from and open subset of S in a open subset $\eta(U) \subset \mathbf{R}^n$. It can be written

$$\xi(P) = (x^1(p), \dots, x^n(p)), \qquad \forall p \in U$$

where the maps x^i are called the *coordinates functions* of ξ and we will denote $\xi = (x^1, \dots, x^n)$. Two charts ξ and η of dimension n *intersect in a smooth manner* if the maps $\xi \circ \eta^{-1}$ and $\eta \circ \xi^{-1}$ are of classe $\mathscr{C}^\infty$.

An *atlas* is a collection of charts of dimension n such that

- for all point $p \in S$, there exist an open subset U such that $p \in U$;
- two charts intersect in a smooth manner.

An atlas is *complete* if it contains all the charts of S which intersect in a smooth manner. Any atlas admits a completion.

Définition 1.1. A *differentiable manifold* is a topological space equipped with a complete atlas.

Exemples.
- The euclidean space is a differentiable manifold.
- The sphere $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ is a differentiable manifold of dimension n.
- A cartesian product of differentiable manifolds is also a differentiable manifold.

Définition 1.2. Let M be a differentiable manifold. A function $f\colon M \longrightarrow \mathbf{R}$ is of class $\mathscr{C}^\infty$ if, for any chart (U, η), the maps

$$f \circ \eta^{-1} \colon \eta^{-1}(U) \longrightarrow \mathbf{R}$$

is of class $\mathscr{C}^\infty$.

The sum, product and inverse are of class $\mathscr{C}^\infty$.

Définition 1.3. Let M and N be two differentiable manifolds. A map $\phi\colon M \longrightarrow N$ is of class $\mathscr{C}^\infty$ if, for any charts (U, ξ) of M and (V, η) of N, the map

$$\eta \circ \phi \circ \xi^{-1} \colon \xi(U) \longrightarrow \eta(V).$$

is of class $\mathscr{C}^\infty$.

1.2. Tangent spaces and tangent bundle

Définition 1.4. Let $p \in M$ a point. Let $F(M)$ be the space of functions of class $\mathscr{C}^\infty$ on M. A *tangent vector* at the point p is a $\mathbf{R}$-linear map $v\colon F(M) \longrightarrow \mathbf{R}$ satisfying the Leibniz rules

$$v(fg) = f(p)v(g) + g(p)v(f).$$

The space of all tangent vectors at the point p is the *tangent space at the point* p, denoted $\mathrm{T}_p M$.

Let (U, ξ) a chart, $p \in U$ a point and $f \in F(M)$ a function. We denote $\eta = (x^1, \dots, x^m)$ and

$$\frac{\partial f}{\partial x_i}(p) \coloneqq \frac{\partial (f \circ \eta^{-1})}{\partial u^i}(\eta(p)).$$

where the notation u^i are the coordinates on $\mathbf{R}^m$. The map

$$\partial_i|_p \coloneqq \left.\frac{\partial}{\partial x_i}\right|_p : F(M) \longrightarrow \mathbf{R}$$

are a tangent vector at the point p. The vectors $\partial_i|_p$ form a basis of $\mathrm{T}_p M$.

Définition 1.5. Let $\phi\colon M \longrightarrow N$ be a map of class $\mathscr{C}^\infty$. For all point $p \in M$, we define the $\mathbf{R}$-linear map

$$d\phi_p\colon \mathrm{T}_p M \longrightarrow \mathrm{T}_{\phi(p)} N$$

by the equality

$$d\phi_p(v) = v_\phi \in \mathrm{T}_{\phi(p)} N$$

where

$$v_\phi(g) \coloneqq v(g \circ \phi).$$

With coordinate $(x^1, \dots, x^m)$ on M and $(y^1, \dots, y^n)$, we have

$$d\phi(p)(\partial_j|_p) = \sum_{i=1}^n \frac{\partial (y^i \circ \phi)}{\partial x^j}(p) \left.\frac{\partial}{\partial y^i}\right|_{\phi(p)}.$$

Remarque. If the maps $\phi\colon M \longrightarrow N$ and $\psi\colon N \longrightarrow P$ are smooth, then the composition $\psi \circ \phi$ is also a smooth map.

Définition 1.6. A *vector field* is a a map V which send each point $p \in M$ on a tangent vector $V_p \in \mathrm{T}_p M$.

If $f \in F(M)$, we denote $V(f)(p) \coloneqq V_p(f)$.

Définition 1.7. If $V(f)$ is of class $\mathscr{C}^\infty$ for all $f \in F(M)$, then V is of class $\mathscr{C}^\infty$.

The sum of two vector fields is a vector field. The multiplication of a vector field by a function is a vector field. The *bracket* of two vector fields V and W is defined by

$$[V, W]_p(f) \coloneqq V_p(W(f)) - W_p(V(f)).$$

It is skew-symmectric $\mathbf{R}$-bilinear. It satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We have

$$[fX, gY] = fg[X, Y] + f(X(g))Y - g(Y(f))X.$$

Exemple. We have $[\partial_i, \partial_j] = 0$.

Définition 1.8. A differentiable manifold P is a *sub-manifold* of M if

- $P \subset M$;
- the injection map $j\colon P \hookrightarrow M$ is a map of class $\mathscr{C}^\infty$;
- it differential $dj_p\colon \mathrm{T}_p P \longrightarrow \mathrm{T}_{j(p)} M$ is injective for all $p \in P$.

Théorème 1.9 *(Whitney).* Let M be a $\mathscr{C}^\infty$-differentiable manifold of dimension n. Then there exists an immersion $M \longrightarrow \mathbf{R}^{2n}$.

The *tangent bundle* of M is $\mathrm{T}M := \bigsqcup_{p\in M} \mathrm{T}_p M$. With $\mathrm{T}M = \{(p,v) \mid p \in M, v \in \mathrm{T}_p M\}$, we have a natural map $\pi\colon \mathrm{T}M \longrightarrow M$ which satisfies $\pi^{-1}(p) = \mathrm{T}_p M$. We can show that the tangent bundle is a manifold of dimension $2n$. Indeed, let be (U, ξ) a chart on M with $\xi = (x^1, \dots, x^n)$. Let $v \in \mathrm{T}_p M$. We can write

$$v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p$$

with $v^i \in \mathbf{R}$. We consider

$$\tilde{\eta}\colon \left| \begin{aligned} \pi^{-1}(U) \subset \mathrm{T}M &\longrightarrow \mathbf{R}^{2n}, \\ (p,u) &\longmapsto (x^1(\pi(p,u)), \dots, x^n(\pi(p,u)), v^1, \dots, v^n). \end{aligned} \right.$$

where $v^i = v(x^i) =: \dot{x}^i(u)$. This defines a atlas on $\mathrm{T}M$. If $(u^1, \dots, u^{2n})$ are the coordinates on $\mathbf{R}^{2n}$, the transition functions are given by

$$u^i \tilde{\xi} \circ \tilde{\eta}^{-1} = x^i \circ \pi \circ \tilde{\eta}^{-1}(a,b) = x^i \eta^{-1}(a),$$

$$u^i \tilde{\xi} \circ \tilde{\eta}^{-1} = \dot{x}^i \circ \tilde{\eta}^{-1}(a,b) = \sum b^k \frac{\partial x^i}{\partial y^k}(\eta^{-1}(a)).$$

So the map $\tilde{\xi} \circ \tilde{\eta}^{-1}$ are of class $\mathscr{C}^\infty$.

Remarque. A vector field $X\colon M \longrightarrow \mathrm{T}M$ is a map of class $\mathscr{C}^\infty$ such that $\pi \circ X = \mathrm{Id}_M$.

Remarque. In general, we have $\mathrm{T}M \neq M \times \mathbf{R}^n$. This is the case for $\mathbf{S}^3$.

Exemple. We consider the sphere $\mathbf{S}^2$. It is a manifold of dimension 2. We want to calculate it tangent space at a point $p \in \mathbf{S}^2$. Let $\gamma\colon \left]-\varepsilon, \varepsilon\right[\longrightarrow \mathbf{S}^2$ be a curve of class $\mathscr{C}^\infty$ on $\mathbf{S}^2$ with $\gamma(0) = p$. It acts on functions on $\mathbf{S}^2$. For a function $f\colon \mathbf{S}^2 \longrightarrow \mathbf{R}$, we denote

$$\dot{\gamma}(0) \cdot f := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \gamma)(t)$$

We have

$$\left.\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|_{t=0} = \dot{\gamma}(0) \in \mathrm{T}_p \mathbf{S}^2$$

and all tangent vector can be obtained this way. As $|\gamma| = 1$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} |\gamma(t)|^2 = 0 = 2\langle p, \dot{\gamma}(0)\rangle.$$

Thus we conclude

$$\mathrm{T}_p \mathbf{S}^2 = \{X \in \mathbf{R}^3 \mid \langle X, p\rangle = 0\}.$$

Exemples. Open subsets of $\mathbf{R}^n$ are differentiable manifolds. The half-plane $\mathbf{H}^2 := \mathbf{R} \times \mathbf{R}_+^*$ has the tangent space $\mathrm{T}_p \mathbf{H}^2 = \mathbf{R}^2$ and so its tangle bundle is $\mathrm{T}\mathbf{H}^2 = \mathbf{H}^2 \times \mathbf{R}^2$.

Exemples.
- The image of the map

  $$\left| \begin{aligned} \left]-1, 1\right[&\longrightarrow \mathbf{R}^2, \\ t &\longmapsto (t, |t|) \end{aligned} \right.$$

  is a differentiable manifold but not a submanifold of $\mathbf{R}^2$.
- The map

  $$\left| \begin{aligned} \mathbf{R} &\longrightarrow \mathbf{R}^2, \\ t &\longmapsto (t^3, t^2) \end{aligned} \right.$$

  is differentiable but not an immersion.

– The map
$$\left|\begin{aligned}\mathbf{R} &\longrightarrow \mathbf{R}^2,\\ t &\longmapsto (t^3-4t, t^2-4)\end{aligned}\right.$$
is differentiable and an immersion, but there is a self-intersection sot it is not an embedding.
– The map $t \longmapsto (t, \sin(1/t))$ is an immersion with no self-intersecting point, but it is not an embedding.
– The cone $\left\{x^2+y^2-z^2=0\right\}$ is not a submanifold of $\mathbf{R}^3$ for connectivity reasons.

1.3. Tensors

Let V be a module over a ring K.

Définition 1.10. Let $r, s \in \mathbf{N}$ be integers with $rs > 0$. A *tensor of type* (r,s) is a K-multilinear function
$$(V^*)^r \times V^s \longrightarrow K.$$
We denote $\mathrm{T}^{r,s}(V)$ the set of tensors of type (r,s).

A *tensor field* is a tensor on the ring $\mathscr{X}(M)$ which denotes the set of vectors field on a differentiable manifold M. The set $\mathscr{X}(M)$ is a module on the ring $\mathrm{F}(M)$ of functions on M. So a tensor field of type (r,s) is a $\mathrm{F}(M)$-linear map
$$A\colon \mathscr{X}^*(M)^r \times \mathscr{X}(M)^s \longrightarrow \mathrm{F}(M).$$

Exemple. The map
$$C\colon \left|\begin{aligned}\mathscr{X}^*(M) \times \mathscr{X}(M) &\longrightarrow \mathrm{F}(M),\\ (\theta, X) &\longmapsto \theta(X)\end{aligned}\right.$$
is a tensor.

Counter-example. Let $\omega \in \mathscr{X}^*(M)$ a linear form. The map
$$F\colon \left|\begin{aligned}\mathscr{X}(M) \times \mathscr{X}(M) &\longrightarrow \mathrm{F}(M),\\ (X, Y) &\longmapsto X(\omega(Y))\end{aligned}\right.$$
is not a tensor field.

Remarque. When $A \in \mathrm{T}^{r,s}(V)$ and $B \in \mathrm{T}^{r',s'}(V)$, we can define the tensor $A \times B \in \mathrm{T}^{r+r',s+s'}(V)$ with the equality
$$A \otimes B(\theta^1, \dots, \theta^{r+r'}, X_1, \dots, X_{s+s'}) = A(\theta^1, \dots, \theta^r, X_1, \dots, X_s) A(\theta^{r+1}, \dots, \theta^{r+r'}, X_{s+1}, \dots, X_{s+s'}).$$

Proposition 1.11. Let $p \in M$ and $A \in \mathrm{T}^{r,s}(M)$. Let $\overline{\theta}^i$ and θ^i be 1-forms which agree on p. Let $\overline{X}_i$ and X_i be vector field which agree on p. Then
$$A(\overline{\theta}^1, \dots, \overline{\theta}^r, \overline{X}_1, \dots, \overline{X}_s)(p) = A(\theta^1, \dots, \theta^r, X_1, \dots, X_s)(p).$$

Thus we can define the map
$$A_p\colon (\mathrm{T}_p^* M)^r \times (\mathrm{T}_p M)^s \longrightarrow \mathbf{R}.$$

Démonstration. We show that, if $\theta^{i_0}(p) = 0$ or $X_{i_0}(p) = 0$, then $A(\theta^1, \dots, \theta^r, X_1, \dots, X_s)(p) = 0$. Let $(U, (x^1, \dots, x^n))$ be a chart. Then we can write $X_{j_0} = \sum X^i \partial_i$. Let f be a bump function on U with $f(p) = 1$. We have $X_{j_0}(p) = 0 \Leftrightarrow X^i(p) = 0, \forall i$ and $f^2 X_{j_0}$ is a vector field and we can write $f^2 X_{j_0} = \sum f X^i (f \partial_i)$. So
$$\begin{aligned} f^2 A(\theta^1, \dots, \theta^r, X_1, \dots, X_s) &= A(\theta^1, \dots, \theta^r, X_1, \dots, f^2 X_{j_0}, \dots, X_s)\\ &= \sum_i f x^I A(\theta_1, \dots, \theta^r, X_1, \dots, f\partial_i, \dots, X^s)\end{aligned}$$
and $A(\theta^1, \dots, \theta^r, X_1, \dots, X_s)(p) = 0$. ◇

Let $(U,(x^1,\dots,x^n))$ be a map. Let $p\in U$. On U, we denote

$$A^{i_1,\dots,i_s}_{j_1,\dots,j_s} \coloneqq A(\mathrm{d}x^{i_1},\dots,\mathrm{d}x^{i_r},\partial j_i,\dots,\partial j_s)$$

and we have

$$A=\sum A^{i_1,\dots,i_s}_{j_1,\dots,j_s}\partial j_i\otimes\cdots\otimes\partial j_s\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_r}.$$

The *contraction* of A on the indices i and j is the tensor field C^i_jA of type $(r-1,s-1)$ which is the composition of C and the tensor

$$(\theta,X)\longmapsto A(\theta^1,\dots,\theta,\dots,\theta^r,X_1,\dots,X,\dots,X_s).$$

The component of C^i_jA are $A^{i_1,\dots,m,\dots,i_r}_{j_1,\dots,m,\dots,j_s}$ with $m\in\{1,\dots,n\}$.

Définition 1.12. Let $\phi\colon M\longrightarrow N$ a differentiable map. If $A\in\mathrm{T}^{0,s}(N)$, we set

$$\phi^*A(X_1,\dots,X_s)\coloneqq A(d\phi(X_1),\dots,d\phi(X_s)).$$

The tensor $\phi^*A\in\mathrm{T}^{0,s}(M)$ is the *pull-back of A by ϕ.*

Définition 1.13. A *derivation of tensor* is a $\mathbf{R}$-linear map

$$D\colon \mathrm{T}^{r,s}(M)\longrightarrow\mathrm{T}^{r,s}(M)$$

such that

$$D(A\otimes B)=DA\otimes B+A\otimes DB$$

and

$$D(CA)=C(DA).$$

For a function $f\in\mathrm{F}(M)\subset\mathrm{T}^{0,0}(M)$, we set $f\otimes A=fA$ and we have $D(fA)=fDA+(Df)A$. The derivation D is a derivation of functions so there exists a $V\in\mathscr{X}(M)$ such that $Df=V(f)$. The chain rule becomes

$$\begin{aligned}D(A(\theta^1,\dots,\theta^r,X_1,\dots,X_s))&=(DA)(\theta^1,\dots,\theta^r,X_1,\dots,X_s)\\&\quad+\sum_{i=1}^{r}A((\theta^1,\dots,D\theta^i,\dots,\theta^r,X_1,\dots,X_s))+\sum_{i=1}^{s}(\theta^1,\dots,\theta^r,X_1,\dots,DX_j,\dots,X_s).\end{aligned}$$

Théorème 1.14. Given a vector field V and an $\mathbf{R}$-linear map $\delta\colon\mathscr{X}(M)\longrightarrow\mathscr{X}(M)$ such that

$$\delta(fX)=V(f)W+f(\delta X),$$

there exists a unique derivation of tensors which equals to δ on $\mathscr{X}(M)$ and V on $\mathrm{F}(M)$.

Définition 1.15. Let $V\in\mathscr{X}(M)$. Then we set the derivation L_V as

$$L_V(f)\coloneqq V(f)\qquad\text{et}\qquad L_V(X)\coloneqq[V,X]$$

for all $f\in\mathrm{F}(M)$ and $X\in\mathscr{X}(M)$. It is called the *Lie's derivation.*

Définition 1.16. Let V be a vector space. The *index* of a bilinear form b is the dimension of the largest subspace $W\subset V$ such that the restriction $b|_{W\times W}$ is negative definite.

A vector $v\in V$ is *null* or *isotropic* if $v\neq0$ and $b(v,v)=0$.

Lemme 1.17. Let V and W be two linear spaces of the same dimension. Then they are equipped with inner products with the same indices if and only if the exists a linear isometry $V\longrightarrow W$.

Chapitre 2
Semi-riemannian manifolds

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2.1. First definitions

Définition 2.1. A *metric* on a differentiable manifold M is a tensor field g on M of type $(0,2)$ which is symmetric, non-degenerate and with a constant index. A *semi-riemannian manifold* is a manifold M equipped with a metric g.

In general, two different metrics on a same manifold M gives two different semi-riemannian structures on M. If the index is zero, then we say that the semi-riemannian manifold (M,g) is *riemannian*. If the index is one, then we will call it *lorentzian*.

In local coordinates $(U,(x^1,\dots,x^n))$, we can write $g=\sum g_{i,j}\,\mathrm{d}x^i\otimes\mathrm{d}x^j$ with $g_{i,j}=g(\partial i,\partial j)$. The matrix $(g_{i,j})$ is invertible, the inverse will be denoted $(g^{i,j})$.

Exemple. Let $\nu\leqslant n$ be a natural integer. On the space $\mathbf{R}^n$, we have the semi-riemannian structure $\mathbf{R}^n_\nu$ with the metric

$$\langle u,v\rangle=-\sum_{i=1}^{\nu}u^iw^i+\sum_{i=\nu+1}^{n}v^iw^i.$$

Définition 2.2. Let $p\in M$. Let (M,g) be a semi-riemannian manifold. A tangent vector $v\in\mathrm{T}_pM$ is

- *space-like* if $v=0$ or $g(v,v)=0$;
- *null* if $v\neq 0$ and $g(v,v)=0$;
- *time-like* if $g(u,u)<0$.

Null vectors form the *null cone*.

If $P\subset M$ is a submanifold and M is equipped with a riemannian metric g, the P is a riemannian manifold. For example, the sphere $\mathbf{S}^2$ admits a riemannian metric. But this is not always true for semi-riemannian metrics.

Lemme 2.3. Let (M,g_M) and (N,g_N) two semi-riemannian manifolds. Let $\pi\colon M\times N\longrightarrow M$ and $\sigma\colon M\times N\longrightarrow N$ the two projections. Then the map $g:=\pi^*g_N+\sigma^*g_N$ is a semi-riemannian metric on $M\times N$.

Définition 2.4. An *isometry* between two semi-riemannian manifolds (M,g) and (N,h) is a diffeomorphism $\phi\colon M\longrightarrow N$ which preserves the metrics, that is $\phi^*g=h$ or

$$\forall p\in M,\ \forall u,w\in\mathrm{T}_pM,\qquad h_{\phi(p)}(d\phi_p(u),d\phi_p(v))=g_p(u,v).$$

Exemple. Let $(u^1, \dots, u^n)$ be the natural coordinates on $\mathbf{R}^n_\nu$. Let V and W two vector fields on $\mathbf{R}^n$. We denote $W = \sum W^i \partial_i$. We define $D_V W \coloneqq dW(V) = \sum V(W^i)\partial_i$. This is the *covariant derivative* of W with respect to V.

2.2. Connection and Levi-Civita connection

Définition 2.5. A *connection* on a manifold M is a map

$$D \colon \left| \begin{aligned} \mathscr{X}(M) \times \mathscr{X}(M) &\longrightarrow \mathscr{X}(M), \\ (X, Y) &\longmapsto D_X Y \end{aligned} \right.$$

such that

- $D_V W$ is $\mathrm{F}(M)$-linear in V ;
- $D_V W$ is $\mathbf{R}$-linear in W ;
- $D_V(fW) = V(f)W + fD_V W$.

Proposition 2.6. Let (M, g) be a semi-riemannian manifold. Let $V \in \mathscr{X}(M)$ a vector field. Let V^* be the 1-form defined by

$$V^*(X) \coloneqq g(V, X).$$

Then the map $V \longmapsto V^*$ is a $\mathrm{F}(M)$-linear isomorphism.

Exemples.
- We take the sphere $\mathbf{S}^2$. For a point $p \in \mathbf{S}^2$ and two tangent vectors $X, Y \in \mathrm{T}_p\mathbf{S}^2$, we can define

$$g_p(X, Y) \coloneqq \langle X, Y \rangle$$

where the notation $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbf{R}^3$. This gives a riemannian metric on $\mathbf{S}^2$. If we replace the inner product $\langle \cdot, \cdot \rangle$ by another semi-riemannian metric on $\mathbf{R}^3$, then this metric is no longer semi-riemannian in general.
- If g is a riemannian metric, then a another riemannian metric is given by

$$\tilde{g}_p(X, Y) \coloneqq e^{f(p)} g_p(x, y)$$

for a smooth function $f \in \mathscr{C}^\infty(M, \mathbf{R})$.
- There exists three vector fields E_i on $\mathbf{S}^3$ which form a orthonormal family where the semi-riemannian is the same as the first example. Then we can define a new semi-riemannian metric by
  - $\langle E_i, E_j \rangle = 0$ for all $i \neq j$;
  - $|E_1|^2 = -1$ and $|E_2|^2 = |E_3|^2 = 1$.
- On the half-plane $\mathbf{H}^2$, we can define the metric

$$g \coloneqq \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

Question. When can we equip M with a semi-riemannian metric ? It is not always the case for a semi-riemannian metric with strictly positive index. But it is always the cases for a riemannian metric. There is two ways to do that :

- by using the Withney's theorem : the exists an immersion $\iota \colon M \longrightarrow \mathbf{R}^N$ for a large enough integer N and we take the pullback of the euclidean matric on $\mathbf{R}^N$, that is

$$g_p(X, Y) \coloneqq \langle d\iota_p(X), d\iota_p(Y) \rangle;$$

- if (U_i, x^i) are an atlas of M, we define

$$g_p \coloneqq \sum_i \alpha_i x_i^* \langle \cdot, \cdot \rangle_{\mathbf{R}^n}.$$

Théorème 2.7. Let (M, g) be a semi-riemannian manifold. Then there exists a unique connection D such that, for all vector fields V and W, we have

- $[V, W] = D_V W - D_W V$;
- $Xg(V, W) = g(D_X V, W) + g(V, D_X W)$

Moreover, the connection D is characterized by the *Koszul formula*

$$2g(D_V W, X) = V(g(X, W)) + W(g(X, V)) - Xg(V, W) \\ - g(V, [W, X]) + g(W, [X, V]) + g(X, [V, W]).$$

It is called the *Levi-Civita connection.*

Démonstration. Let D be a connection satisfying these two points. In the right-hand side of the Koszul formula, using the two points, we obtain $2g(D_V W, X)$. This proves the uniqueness because of the one-to-one correspondance between vector fields and 1-forms.

Let proves the existence. Let $F(V, W, X)$ the right-hand side of the Koszul formula. Then if we take two vector fields V and W, then the map $F(V, W, \cdot)\colon \mathscr{X}(M) \longrightarrow \mathbf{R}$ is $\mathrm{F}(M)$-linear. So it is a 1-form. Thus there exists a unique vector field $D_V W$ such that

$$g(D_V W, X) = F(V, W, X), \qquad \forall X \in \mathscr{X}(M).$$

This show the Koszul formula and that the map D is a connection. With this formula, we can prove the two points. ◇

Notation. We will write ∇ for the Levi-Civita connection. With this notation and $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$, the two points of the theorem are

$$X\langle Y, Z\rangle = \langle \nabla_X Z, Y\rangle + \langle \nabla_X Y, Z\rangle, \\ [X, Y] = \nabla_X Y - \nabla_Y X.$$

Définition 2.8. The *Christoffel symbols* for the chart (U, x^i) are the functions on U given by

$$D_{\partial_i}(\partial_j) = \sum_{k=1}^{n} \Gamma_{i,j}^k \partial_k.$$

Recall that $[\partial_i, \partial_j] = 0 = D_{\partial_i}(\partial_j) - D_{\partial_j}(\partial_i)$ by the Schwarz theorem, so $\Gamma_{i,j}^k = \Gamma_{j,i}^k$. Moreover, if $W = \sum W^j \partial_j$ on U, then

$$\begin{aligned} D_{\partial_i}(W) &= \sum_j (\partial_i(W^j)\partial_j + W^j \sum_k \Gamma_{i,j}^k \partial_k) \\ &= \sum_k (\partial_i(W^j) + \sum_j W^j \Gamma_{i,j}^k)\partial_k \end{aligned}$$

By Koszul formula, we have

$$\Gamma_{i,j}^k = \frac{1}{2} \sum_\ell g^{k,\ell} \left(\frac{\partial g_{\ell,j}}{\partial x^i} + \frac{\partial g_{\ell,i}}{\partial x^j} - \frac{\partial g_{i,j}}{\partial x^\ell} \right).$$

Exemple. On $\mathbf{R}_\nu^n$, we have $\Gamma_{i,j}^k = 0$.

2.3. Curvature and Ricci tensor

Definition-proposition 2.9. Let (M, g) a semi-riemannian manifold and ∇ its Levi-Civita connection. The the map

$$R\colon \left| \begin{aligned} &\mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \longrightarrow \mathscr{X}(M) \\ &\qquad (X, Y, Z) \longmapsto R(X, Y)Z := \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z \end{aligned} \right.$$

is a tensor field of type $(1, 3)$, called the *riemannian curvature.*

There is a version of type $(0, 4)$ given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Démonstration. We need to check that $R(fX,Y)Z = fR(X,Y)Z$ and $R(X,Y)(fZ) = fR(X,Y)Z$. We have $[fX,Y] = fXY - Y(f)X - fYX$ and

$$\nabla_{[fX,Y]}Z = f\nabla_X\nabla_Y Z$$

which prove the formula. ◇

Proposition 2.10. We have the following properties :

1. $R(X,Y)Z = -R(X,Y)Z$;
2. $g(R(X,Y)Z,W) = -g(R(X,Y)W,Z)$;
3. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$;
4. $g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$.

Démonstration. 2. With $g(\cdot,\cdot) = \langle\cdot,\cdot\rangle$, one has

$$\begin{aligned} g(R(X,Y)Z,Z) &= \langle \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z, Z\rangle \\ &= \langle \nabla_{[X,Y]}Z, Z\rangle - \langle \nabla_X\nabla_Y Z, Z\rangle + \langle \nabla_Y\nabla_X Z, Z\rangle \\ &= [X,Y]\left(\frac{\langle Z,Z\rangle}{Z}\right) - X\langle \nabla_Y Z, Z\rangle + \langle \nabla_Y Z, \nabla_X Z\rangle + Y\langle \nabla_X Z, Z\rangle - \langle \nabla_X Z, \nabla_Y Z\rangle \\ &= [X,Y]\left(\frac{\langle Z,Z\rangle}{Z}\right) - XY\left(\frac{\langle Z,Z\rangle}{Z}\right) + Y\left(X\left(\frac{\langle Z,Z\rangle}{Z}\right)\right) = 0. \end{aligned}$$

So $g(R(X,Y)(Z+W), Z+W) = 0$ and we conclure by bilinearity. ◇

Remarque. The map R is a tensor. For $X,Y,Z \in \mathscr{X}(M)$ and $p \in M$, the quantity $(R(X,Y)Z)_p$ only depend on the values $X(p)$, $Y(p)$ and $Z(p)$. So we can define $R_p(u,v)w$ for $u,v,w \in \mathrm{T}_pM$.

Proposition 2.11. Let X, Y and Z be three vector fields. Then

$$(\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) = 0.$$

Remarque. We have

$$(\nabla_X R)(Y,Z)W = \nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W.$$

Moreover, we have

$$(\nabla_X g)(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y,Z) - g(X,\nabla_X Z) = 0.$$

Démonstration. We prove the identity on a basis. We choose $X = \partial_i$, $Y = \partial_j$ and $Z = \partial_k$. So

$$(\nabla_Z R)(X,Y)W = [\nabla_Z, R(X,Y)]W - R(\nabla_Z X,Y)W - R(X,\nabla_Z Y)W$$

and then

$$\begin{aligned} (*) &= (\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) \\ &= [\nabla_Z, R(X,Y)]W + [\nabla_X, R(Y,Z)]W + [\nabla_Y, R(Z,X)] \\ &\qquad - R(\nabla_Z X,Y)W - R(X,\nabla_Z Y)W \\ &\qquad - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z)W \\ &\qquad - R(\nabla_Y Z,X)W - R(Z,\nabla_Y X)W \\ &= [\nabla_Z, R(X,Y)]W + [\nabla_X, R(Y,Z)]W + [\nabla_Y, R(Z,X)] \\ &\qquad + R([X,Z],Y)W + R([Z,Y],X)W + R([Y,X],Z)W. \end{aligned}$$

But

$$[\nabla_Z, R(X,Y)] = [\nabla_Z, [\nabla_X,\nabla_Y]] - [\nabla_Z, \nabla_{[X,Y]}]$$

where the last term is null on U and so

$$(*) = [\nabla_Z, [\nabla_X,\nabla_Y]] = [\nabla_X, [\nabla_Y,\nabla_Z]] + [\nabla_Y, [\nabla_Z,\nabla_Y]] = 0.$$ ◇

Let $p \in M$ be a point and Π be a plane in T_pM. For two tangent vectors $v,w \in \mathrm{T}_pM$ not null and not colinear, we define

$$Q(v,w) := \langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2.$$

This gives a semi-riemannian metric. The plane Π is *nondegenerate* if

$$Q(v,w) \neq 0, \qquad \forall v,w \in \Pi \setminus \{0\},\ v \not\propto w.$$

The quantity $|Q(v,w)|$ is the volume of the parallelogram defined by the vectors v and w. If Π is not degenerate, then we define the *sectional curvature* of Π bu

$$K(\Pi) := \frac{g(R(v,w)v,w)}{Q(v,w)}.$$

This definition does not depend on the choice of the vectors v and w.

Proposition 2.12. If $K(\Pi) = 0$ for all plane $\Pi \subset \mathrm{T}_pM$, then $R = 0$ at the point p.

Démonstration. 1. If v and w define a nondegenerate plane Π, then it suffices to apply the implication $K(\Pi) = 0 \Rightarrow \langle R(v,w)v,w\rangle = 0$.

2. If they define a degenerate plane, then v and w can be approximated by vectors which define a nondegenerate plane. If v is null, let x be a tangent vector such that $\langle u,x\rangle \neq 0$. If not, let x be the opposite of the causal type of v. Then $Q(u,x) < 0$. Let $\delta \neq 0$ a small real number such that the vectors v and $w + \delta x$ define a nondegenerate plane. We assume $\delta = 1$. So thank to the first case, we get

$$\langle R(u,w)u,x\rangle + \langle R(u,x)u,w\rangle = 0$$

which implies $\langle R(u,w)u,x\rangle = 0$ for all x and so $R(u,w)u = 0$. Thus $R(v+x,w)(v+x) = 0$ and $R(v,w)x + R(x,w)v = 0$, so $R(u,w)x = R(w,x)u$. If we do $u \longleftrightarrow w$, we get $R(w,u)x = R(u,x)u$. So we have $R(u,w)x = R(w,x)u = R(x,v)w$. But the first Bianchi identity gives

$$R(u,w)x + R(w,x)u + R(x,u)w = 0$$

and so $R(u,w)x = 0$. Thus $R = 0$.

◇

Corollaire 2.13. If the sectional curvature of M is, at a point p, constant to c, then

$$\forall x,y,z \in \mathrm{T}_pM, \qquad R(x,y)z = c(\langle y,z\rangle x - \langle x,z\rangle y).$$

Définition 2.14.

- The *Ricci tensor* is a $(0,2)$-tensor obtainned by contraction of

$$\mathrm{Ric}(X,Y) := \sum_{m=1}^{n} \varepsilon_m g(R(X,E_m),E_m)$$

where (E_i) is a orthonormal frame (whe $g(E_i,E_j) = 0$ if $i \neq j$ and $g(E_i,E_i) = \varepsilon_i = \pm 1$). This defines a symmetric form.

- The *scalar curvature* is the function

$$\mathrm{scal} := \sum_{m=1}^{n} \varepsilon_m \,\mathrm{Ric}(E_m,E_m).$$

Proposition 2.15. We have $d\mathrm{scal} = 2\,\mathrm{div}(\mathrm{Ric})$.

Démonstration. We set the notations.

- If f is a function, then df is a 1-form defined by $df(X) := X(f)$.
- If T is a $(0,2)$-tensor, then $\mathrm{div}\,T$ is a 1-form defined by $\mathrm{div}\,T(X) := \sum_{m=1}^{n} \varepsilon_m \nabla_{E_m} T(E_m,X)$.
- We work with a vector field X such that $(\nabla_Y X)_p = 0$ for $Y \in \mathscr{X}(M)$.
- We work with an orthonormal basis (E_m) such that $(\nabla_Y E_m)_p = 0$ for $Y \in \mathscr{X}(M)$.

At p, we have

$$\begin{aligned} d\mathrm{scal} &= X(\mathrm{scal}) \\ &= \sum_{m,j} \varepsilon_m \varepsilon_j X(g(R(E_m,E_j)E_j,E_m)) \\ &= \sum_{m,j} \varepsilon_m \varepsilon_j [g(\nabla_X(R(E_m,E_j)E_j),E_m) + g(R(E_m,E_j)E_j,\nabla_X E_m)] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m,j} \varepsilon_m \varepsilon_j g((\nabla_X R)(E_m, E_j)E_j + R(\nabla_X E_m, E_j)E_m + R(E_m, \nabla_X E_i)E_m + R(E_m, E_j)\nabla_X E_m, E_m) \\
&= \sum_{m,j} \varepsilon_m \varepsilon_j g((\nabla_X R)(E_m, E_j)E_j, E_m) \\
&= \sum_{m,j} \varepsilon_m \varepsilon_j [-g((\nabla_{E_j} R)(X, E_m)E_j, E_m) - g((\nabla_{E_m} R)(E_j, X)E_j, E_m)] \\
&= -\sum_{m,j} \varepsilon_m \varepsilon_j [(\nabla_{E_m} R)(E_j, X, E_j, E_m) + \nabla_{E_j} R(X, E_m, E_j, E_m)] \\
&= \sum_{m,j} \varepsilon_m \varepsilon_j [(\nabla_{E_j} R)(E_j, E_m, E_m, X) + \nabla_{E_M} R(E_m, E_j, E_j, X)] \\
&= 2\sum_{m,j} \varepsilon_m \varepsilon_j (\nabla_{E_j} R)(E_j, E_m, E_m, X) \\
&= 2\sum_{m,j} \varepsilon_m \varepsilon_j \nabla_{E_j} (R(E_j, E_m, E_m, X)) \qquad \text{because } \nabla g = 0 \\
&= 2\sum_{m,j} \varepsilon_m \varepsilon_j \nabla_{E_j} (\mathrm{Ric}(E_j, X)) \\
&= 2\sum_{m,j} \varepsilon_m \varepsilon_j \nabla_{E_j} (\mathrm{Ric}(X, E_j)) \\
&= 2\sum_{m,j} \varepsilon_m \varepsilon_j (\nabla_{E_j} \mathrm{Ric})(E_j, X) \\
&= 2\,\mathrm{div}(\mathrm{Ric})X.
\end{aligned}
$$

◇

Definition-proposition 2.16. A semi-riemannian manifold (M, g) is an *Einstein manifold* if there exists a function f on M such that

$$\mathrm{Ric}_p = f(p) g_p, \qquad \forall p \in M. \tag{$*$}$$

If the dimension is greater than 3, then the function f is constant.

Démonstration. The idea is to take the divergence of the equation $(*)$. We have

$$
\begin{aligned}
\mathrm{div}(fg)(X) &= \sum_i \varepsilon_i (\nabla_{E_i}(fg))(E_i, X) \\
&= \sum_i \varepsilon_i [E_i(fg(E_i, X)) - fg(\nabla_{E_i} E_i, X) - fg(E_i, \nabla_{E_i} X)] \\
&= \sum_i \varepsilon_i [E_i(f)g(X, E_i) + f(E_i g(X, E_i)) - fg(\nabla_{E_i} E_i, X) - fg(E_i, \nabla_{E_i} X)] \\
&= \sum_i \varepsilon_i E_i(f) g(X, E_i) \\
&= \Big(\sum_i \varepsilon_i g(X, E_i) E_i\Big)(f) \\
&= X(f) = df(X)
\end{aligned}
$$

and so $\mathrm{div}(fg) = df$. Taking the divergence of the equation $(*)$, we obtain $\mathrm{div}(\mathrm{Ric}) = df$ and, by the last proposition, we have $df = d\mathrm{scal}/2$. Taking its trace, we get $\mathrm{scal} = (\dim M) f$. Then we have $\mathrm{scal} = 2f + K$ for a constant K. As $\dim M \geqslant 2$, the function scal is a constant and so does the function f. ◇

2.4. Killing vector field

Définition 2.17. A *Killing vector field* is a vector field such that the Lie derivative of the matrix g with respect to X is zero, that is
$$\mathscr{L}_X\, g = 0.$$

The associated flow of a vector field is the map
$$\Psi : \left| \begin{array}{l} M \times I \longrightarrow M, \\ (p,t) \longmapsto \Psi_t(p) \end{array}\right.$$
such that
$$\Psi(p,0) = p \qquad \text{et} \qquad \left.\frac{\mathrm{d}\Psi(t,p)}{\mathrm{d}t}\right|_{t=0} = X(p).$$

Proposition 2.18. A vector field X is a Killing vector field if and only if its flow is by isometries.

Démonstration. First, we show that
$$\mathscr{L}_X\, g = \lim_{t\longrightarrow 0} \frac{1}{t}[\Psi_t^* g - g]. \tag{1}$$
By the chain rule, one has
$$\begin{aligned}(\mathscr{L}_X\, g)(A,B) &= X(g(A,B)) - g(\mathscr{L}_X\, A, B) - g(A, \mathscr{L}_X\, B)\\ &= X(g(A,B)) - g([X,A],B) - g(A,[X,B])\\ &= g(\nabla_X A, B) + g(A, \nabla_X B) - g(\nabla_X A - \nabla_A X, B) - g(A, \nabla_X B - \nabla_B X)\\ &= g(\nabla_A X, B) + g(\nabla_B X, A).\end{aligned}$$
So X is a Killing vector field if and only if
$$\forall A, B, \qquad g(\nabla_A X, B) = -g(A, \nabla_B X).$$
But we have
$$\begin{aligned}(\Psi_t^* g - g)(A,B) &= g(d\Psi_t(A), d\Psi_t(B)) - g(A,B)\\ &= g(d\Psi_t(A), d\Psi_t(B)) - g(A_{\Psi_t}, B_{\Psi_t}) + g(A_{\Psi_t}, B_{\Psi_t}) - g(A,B).\end{aligned}$$
On the one hand, with $F = G \circ \alpha$, $\alpha(t) = \Psi_t$ and $G = g(A,B)$, we get
$$\begin{aligned}\lim_{t\longrightarrow 0}[g(A_{\Psi_t}, B_{\Psi_t}) - g(A,B)] &= F'(0)\\ &= Xg(A,B).\end{aligned}$$
On the other hand, with $\tilde{A} = A_{\Psi_t}$ and $A \longleftrightarrow A_{\Psi_t}$, we have
$$\begin{aligned}g(d\Psi_t(A), d\Psi_t(B)) - g(A_{\Psi_t}, B_{\Psi_t}) &= g(\tilde{A}, \tilde{B}) - g(A,B)\\ &= g(\tilde{A} - A, \tilde{B}) + g(A, \tilde{B} - B)\end{aligned}$$
with
$$\begin{aligned}\lim_{t\longrightarrow 0} g(\tilde{A} - A, \tilde{B}) &= \lim_{t\longrightarrow 0} \frac{1}{t}[g(d\Psi_t(A) - A_{\Psi_t}, d\Psi_t(B))]\\ &= \lim_{t\longrightarrow 0} \frac{1}{t} g(d\Psi_t(A - d\Psi_{-t}(A_{\Psi(t)})), d\Psi_t(B))\\ &= -\lim_{t\longrightarrow 0} \frac{1}{t} g(d\Psi_t(d\Psi_{-t}(A_{\Psi(t)}) - A), d\Psi_t(B))\\ &= -g(\lim_{t\longrightarrow 0} \frac{1}{t}[d\Psi_t(d\Psi_{-t}(A_{\Psi(t)}) - A)], \lim_{t\longrightarrow 0} d\Psi_t(B_{\Psi_t})).\end{aligned}$$
But $\lim_{t\longrightarrow 0} d\Psi_t(B_{\Psi_t}) = B$ and
$$\lim_{t\longrightarrow 0} \frac{1}{t}[d\Psi_t(d\Psi_{-t}(A_{\Psi(t)}) - A)] = -[A,X]. \tag{$*$}$$
If we conclude the equality $(*)$, then we will get the formula (1). Let proove the equality $(*)$.

Let Ψ be the flow of a vector field V. We must prove

$$[V,W] = \lim_{t\longrightarrow 0}\frac{1}{t}[d\Psi_{-t}(W_{\Psi_t} - W)] \tag{2}$$

Let $F_p(t) = d\Psi_{-t}(W_{\Psi_t(p)}$. The right-hand side of the equation (2) is exactly $F_p'(0)$. Assume $V_p \neq 0$. Let (x_i) a system of local coordinates such that $\partial/\partial x_i = V$. Then locally

$$x^1(\Psi_t(q)) = x^1(q) + t \qquad \text{et} \qquad x^j(\Psi_t(q)) = x^j(q), \qquad j \geqslant 2$$

and

$$d\Psi_t\left(\frac{\partial}{\partial x_i}\right) = \sum_j \frac{\partial \Psi^j}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_i}(\Psi_t).$$

Let $W = \sum_i W^i \frac{\partial}{\partial x_i}$. Thus

$$F_p(t) = \sum_i W^i(\Psi_t(p))\frac{\partial}{\partial x_i}.$$

and so

$$\begin{aligned} F_p'(0) &= \sum_i \frac{\mathrm{d}(W^i \circ \Psi_t(p))}{\mathrm{d}t}\bigg|_{t=0}\frac{\partial}{\partial x_i} \\ &= \sum_i V_p(W^i)\frac{\partial}{\partial x_i} \\ &= \sum_i \frac{\partial W^i}{\partial x^1}\partial_i. \end{aligned}$$

So

$$\begin{aligned} [V,W] &= (\partial_1, W) \\ &= [\partial_1, \sum W^i \partial_i] \\ &= \sum \partial_1(W^1)\partial_i + W^i[\partial_1, \partial_i] \\ &= \sum \frac{\partial X^i}{\partial x^1}\partial_i = F_p'(0). \end{aligned}$$

◇

Lemme 2.19. Let X be a Killing vector field on a connected manifold M such that, for all point $p \in M$, we have

$$X(p) = 0 \qquad \text{et} \qquad (\nabla X)_p = 0.$$

Then $X = 0$ on M.

Démonstration. The set

$$A := \{q \in M \mid X(q) = 0 \text{ et } (\nabla X)_q = 0\}$$

is closed and nonempty. To conclude, we show that this set is open. Take $p \in A$ et Ψ_t the associated flow.

Since $X(p) = 0$, we have $\Psi_t(p) = 0$ for all t. Indeed, the flow satisfies $\Psi_t \circ \Psi_s = \Psi_{t+s}$. Thus we get

$$\frac{\mathrm{d}\Psi_{t+s}}{\mathrm{d}t}\bigg|_{t=0}(p) = \frac{\mathrm{d}\Psi_t}{\mathrm{d}t}\bigg|_{t=0}(p) = X(p) = 0$$

which concludes $\Psi_t(p) = \Psi_0(p) = p$.

Let proves that $d\Psi_t \colon \mathrm{T}_pM \longrightarrow \mathrm{T}_pM$ is the identity. We have $[X,Y]_p = (\nabla_X Y)_p - (\nabla_Y X)_p$. But the points $(\nabla_X Y)_p$ depends only on $X(p) = 0$, so $(\nabla_X Y)_p = 0$. So we have $(X(Y) - Y(X))_p = 0$ and

$$\frac{d\Psi_t(Y_{\Psi_t}) - Y_p}{t} = 0.$$

Let $F_p(t) := d\Psi_t(Y_{\Psi_t})$. Then $F_p'(0) = 0$. With $Y = d\Psi_s(u)$, the function $t \longmapsto d\Psi_{-t}(Y_{\Psi_t})$ has a null derivative, so $d\Psi_t = 0$. Finally, the flow acts by isométries, so the flow acts by the identity. ◇

Remarque. Klling vector fields form a Lie algebra of finite dimension because $[\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{L}_{[X,Y]}$. Moreover, the dimension is less then $n(n+1)/2$ where $n = \dim M$.

Proposition 2.20. Let X be a Killing vector field and $f := |X|^2/2$. Then

$$\Delta f = -\operatorname{Ric}(X, X) + |\nabla X|^2.$$

Démonstration. Only calculus. ◇

Remarque. We compute $\operatorname{grad} f = -\nabla_X X$.

Définition 2.21. The *divergence* of a vector field X is

$$\operatorname{div} V := \operatorname{tr}(\nabla V).$$

Théorème 2.22 *(Bochores).* Let M be a compact riemannian manifold with $\operatorname{Ric}(X, X) \leqslant 0$ for all X. Then a Killing vector field is parallel, that is $\nabla X = 0$ on M. If $\operatorname{Ric} < 0$, then there are no nonzero Killing vector field.

Démonstration. But the Stokes theorem, we have

$$\int_M \operatorname{div} V = 0.$$

Let $f = |X|^2/2$. We get

$$\int_M \nabla f = 0 = \int_M -\operatorname{Ric}(Y, X) + |\nabla X|^2$$

and so $\nabla X = 0$. Moreover, if $\operatorname{Ric} < 0$, then $\operatorname{Ric}(Y, X) = 0$ for all Y and so $X = 0$. ◇

Théorème 2.23 *(Berger).* Let M a compact riemannian manifold of even dimension with positive sectional curvature. The any Killing vector field has a zero.

Démonstration. Let X be a vector field. Let $f := |X|^2/2$. Then $\operatorname{grad} f = -\nabla_X X$. If X has no zero, then f has a positive minimum at a point $p \in M$. Then $\operatorname{Hess} f(p) \geqslant 0$. Let V be a vector field. Ten

$$\begin{aligned}\operatorname{Hess} f(V, V) := \langle \nabla_V(\nabla f), V\rangle &= \langle -\nabla_V \nabla_X X, V\rangle \\ &= \langle R(V, X)X, V\rangle + \langle \nabla_V X, \nabla_V X\rangle.\end{aligned}$$

But $B \colon X \longmapsto \nabla_X X$ is skew symmetric, so $(\nabla_X X)(p) = (\operatorname{grad} f)_p = 0$, so B admits $\lambda = 0$ as an eigenvalue with $X(p)$ as a eigenvector. As $\dim M$ is even, there exists another eigenvector V corresponding to $\lambda = 0$. ◇

Chapitre 3
Geodesics

3.1 First definitions 17

3.1. First definitions

Définition 3.1. Let M be a differential manifold. Let $I \coloneqq]-\varepsilon, \varepsilon[\subset \mathbf{R}$ be a interval centered at the origin and $\gamma \colon I \longrightarrow M$ a curve. A *vector field along the curve* γ is a map $X \colon I \longrightarrow \mathrm{T}M$ such that

$$\forall t \in I, \qquad X(t) \in \mathrm{T}_{\gamma(t)}M.$$

Exemple. The map $t \longmapsto (\gamma(t), \gamma'(t))$ is a vector field along the curve γ.

Proposition 3.2. Let M a semi-riemannian manifold et $\gamma \colon I \longrightarrow M$ a curve. Then there exists a unique $\mathbf{R}$-linear operator

$$\frac{D}{\mathrm{d}t} \colon \{\text{vector fields along } \gamma\} \longrightarrow \{\text{vector fields along } \gamma\}$$

such that

- $\frac{D}{\mathrm{d}t}(fX) = \frac{\mathrm{d}f}{\mathrm{d}t}X + f\frac{D}{\mathrm{d}t}X$;
- If $X(t) = Y(\gamma(t))$, then $\frac{D}{\mathrm{d}t}X = (\nabla_\gamma Y) \circ \gamma$.

Démonstration. Let $t_0 \in I$. Let (U, x) a chart on M and $J \subset I$ an interval such that $\gamma(J) \subset U$. Let $X_i \coloneqq \partial/\partial x_i$. If Y is a vector field along γ, we have

$$\mathrm{T}_{\gamma(t)}M \ni Y(t) = \sum_j \alpha_j(t)(X_j)_{\gamma(t)}.$$

With the first two conditions, we get

$$\frac{D}{\mathrm{d}t}Y = \sum_j \alpha_j \frac{D}{\mathrm{d}t}(X_i \circ \gamma) + \sum_k \alpha'_k X_k(\gamma)$$

and, by the third condition, we obtain

$$\dot{\gamma}_t = \sum \dot{\gamma}_i X_i(\circ\gamma)$$

and

$$\frac{D}{\mathrm{d}t}(X_i \circ \gamma) = (\nabla_j X_j) \circ \gamma = \sum_i \dot{\gamma}_i (\nabla_{X_i} X_j) \circ \gamma.$$

Put everything together

$$\frac{D}{\mathrm{d}t}Y = \sum_k \Big(\alpha'_k \sum_{i,j} \Gamma^k_{i,j,\circ\gamma} \gamma'_i \alpha_j\Big) X_k \circ \gamma.$$

Therefore the operator exists and is unique. ◇

Remarque. The quantity $(\nabla_{\gamma'} X)(t)$ depends only on $\dot{\gamma}(t)$. We denote $\frac{D}{\mathrm{d}t}Y$ by $\nabla_{\dot{\gamma}} Y$.

Définition 3.3. Let M be a semi-riemannian manifold and $\gamma\colon I \longrightarrow M$ a curve of class $\mathscr{C}^\infty$. A vector field X along the curve γ is *parallel* if $\nabla_{\dot\gamma} X = 0$.

Définition 3.4. A curve γ is a *geodesic* if $\nabla_{\dot\gamma}\dot\gamma = 0$.

Théorème 3.5. Let M be a semi-riemannian manifold and $\gamma\colon]a,b[\longrightarrow M$ be a curve. Let $t_0 \in I$ be a real number and $X_0 \in \mathrm{T}_{\gamma(t_0)}M$ a tangent vector. Then there exists a unique vector field Y along the curve γ such that $Y(t_0) = X_0$.

Démonstration. Let (U,x) be a chart such that $\gamma(t_0) \in U$. Let $X_i := \partial/\partial x_i$. Let $J \subset I$ be a interval such that $\gamma(J) \subset U$. We denote

$$\dot\gamma(t) = \sum \dot\gamma^i(t) X_i(\gamma(t)) \qquad \text{et} \qquad Y(t) = \sum_j \alpha_j(t) X_j(\gamma(t)).$$

Then

$$\frac{DY}{\mathrm{d}t}(t) = \sum_k \Big[\dot\alpha_k(t) + \sum_{i,j} \alpha_j(t)\dot\gamma_i(t)\Gamma^k_{i,j}(\gamma(t))\Big] X_k(\gamma(t))$$

and so

$$\frac{DY}{\mathrm{d}t}(t) = 0 \quad\Longleftrightarrow\quad \forall k,\ \dot\alpha_k(t) + \sum_{i,j}\alpha_j(t)\dot\gamma_i(t)\Gamma^k_{i,j}(\gamma(t)) = 0. \qquad (*)$$

Let admits the Picard-Lindelöf-Cauchy theorem :

> *Let $f\colon I \times U \longrightarrow \mathbf{R}^n$ a continuous function which is Lipschitz in x. Then there exists an unique solution $x\colon I \longrightarrow \mathbf{R}^n$ of the system*
>
> $$x'(t) = g(t, x(t)) \qquad \textit{et} \qquad x(t_0) = x_0.$$

So there exists a solution to the equation $(*)$ for any initial data. One can extend $Y(t)$ to I because the coefficients in the equation $(*)$ are bounded for $t \in I$. ◇

Lemme 3.6. Let X and Y be two parallel vector field along a curve γ. The the map

$$t \longmapsto g_{\gamma(t)}(X(t), Y(t))$$

a constant. For $X = Y = \dot\gamma$, if γ is a geodesic, then $g(\dot\gamma,\dot\gamma) = |\gamma|^2$ is constant.

Remarque. So causal type of geodesics is preserve on frame (X_i).

Théorème 3.7. Let M be a semi-riemannian manifold. Let $p \in M$ and $v \in \mathrm{T}_pM$. Then there exists an open interval I and a unique geodesic $\gamma\colon I \longrightarrow M$ such that

$$\gamma(0) = p \qquad \text{et} \qquad \dot\gamma(0) = v.$$

Démonstration. Let (U,x) be a chart such that $\gamma(t_0) \in U$. Let $X_i := \partial/\partial x_i$. Let $J \subset I$ be a interval such that $\gamma(J) \subset U$. We write

$$\dot\gamma = \sum_i \dot\gamma_i (X_i \circ \gamma).$$

We have

$$\nabla_{\dot\gamma}\dot\gamma \sum_k \Big[\ddot\gamma_k(t) + \sum_{i,j} \dot\gamma_j(t)\dot\gamma_j(t)\Gamma^k_{i,j} \circ \gamma\Big] X_k(\gamma(t)).$$

So γ is a geodesic if and only if

$$\ddot\gamma_k(t) + \sum_{i,j}\dot\gamma_j(t)\dot\gamma_j(t)\Gamma^k_{i,j}\circ\gamma, \qquad \forall k$$

if on only if its components satisfy the systems of second order nonlinear ordinary differential equation. Existence is given, for any initial data p and v, by the Picard-Lindelöf-Cauchy theorem. ◇

Chapitre 4
Examples

Exemple. The euclidean space $\mathbf{R}^n$ is a semi-riemannian manifold. The geodesics are straight lines. Indeed, we have $\Gamma_{i,j}^k = 0$ and a path γ must verify the equation

$$\ddot{\gamma}^k + \Gamma_{i,j}^k \dot{\gamma}^i \dot{\gamma}_j = 0$$

Exemple. The sphere $\mathbf{S}^n$ is a riemannian manifold. Indeed, it is a differentiable manifolds by the charts

$$\pi_N : \left| \begin{aligned} \mathbf{S}^n \setminus \{N\} &\longrightarrow \mathbf{R}^n, \\ (x_1, \dots, x_n) &\longmapsto \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right) \end{aligned} \right.$$

and

$$\pi_S : \left| \begin{aligned} \mathbf{S}^n \setminus \{S\} &\longrightarrow \mathbf{R}^n, \\ (x_1, \dots, x_n) &\longmapsto \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}} \right) \end{aligned} \right.$$

where the points N and S are the north and south poles. These two charts are bijective and we can verify that there compositions are $\mathscr{C}^\infty$ maps.

We find the tangent spaces. Let $p \in \mathbf{S}^n$. Take a curve $\gamma : \left]-\varepsilon, \varepsilon\right[\longrightarrow \mathbf{S}^n$ with $\gamma(0) = p$. Then we have $|\gamma(0)|^2 = 1$ and thus $\dot{\gamma}(0) \in \mathrm{T}_p\mathbf{S}^n$. We can prove $\mathrm{T}_p\mathbf{S}^n = \{X \in \mathbf{R}^{n+1} \mid \langle p, X \rangle = 0\}$.

We must equip the sphere with a metric. For $X, Y \in \mathrm{T}_p\mathbf{S}^n$, we set

$$g_{\mathbf{S}^n,p}(X, Y) \coloneqq \langle X, Y \rangle_{\mathbf{R}^{n+1}}.$$

Then the tensor g is a metric on the sphere $\mathbf{S}^n$. We get a riemannian manifold.

We must understand the Levi-Civita connection. We define the connection ∇ on $\mathbf{S}^n$ by

$$\nabla_X Y \coloneqq (\partial_X Y)^{\text{tangent}}$$

and we will check that it is indeed the Levi-Civita connection. Here, the « tangent » is the projection on the tangent space according the decomposition $\mathbf{R}^{n+1} = \mathbf{R}p \oplus \mathrm{T}_p\mathbf{S}^n$ and we denote $\partial_X Y = dY(X)$. First, we prove that

$$\nabla_X Y = \partial_X Y + \langle X, Y \rangle p.$$

The normal part of $\partial_X Y$ is $\langle \partial_X Y, p \rangle p$. But $\langle Y, p \rangle = 0$, so $X\langle Y, p \rangle = 0$ and $\langle \partial_X Y, p \rangle + \langle Y, \partial_Y p \rangle = 0$ and $\partial_X p = dp(X) = X$. So the normal part of $\partial_X Y$ is $-\langle X, Y \rangle p$. Next, we observe that

$$\langle Z, \nabla_X Y \rangle = \langle Z, \partial_X Y \rangle.$$

By the Koszul formula, we have

$$2\langle Z, \partial_X Y \rangle = X\langle Z, Y \rangle - Z\langle X, Y \rangle$$

and

$$\langle Z, \partial_X Y \rangle = \langle Z, \nabla_X Y \rangle + Y\langle X, Y \rangle - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle.$$

So we get the Koszul formula. By the uniqueness, this is the Levi-Civita connection.

Let us find the curvature. We have

$$\begin{aligned}-R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \\ &= \nabla_X(\partial_Y Z + \langle Z,Y\rangle p) - \nabla_Y(\partial_X Z + \langle Z,X\rangle p) - (\nabla_{[X,Y]}Z + \langle Z,[X,Y]\rangle p) \\ &= \partial_X\partial_Y z - \partial_Y\partial_X Z - \partial_{[X,Y]}z) + (\langle X,\partial_Y Z\rangle p - \langle Y,\partial_X Z\rangle p - \langle [X,Y],Z\rangle p) + (\partial_X\langle Y,Z\rangle p - \langle X,\langle Y,Z\rangle p\rangle p - \\ &= \langle Y,\partial_Y Z\rangle p - \langle Y,\partial_X Z\rangle p - \langle [X,Y],Z\rangle p + \partial_X(\langle Y,Z\rangle p) - \partial_Y(\langle X,Z\rangle p).\end{aligned}$$

But $\partial_X(\langle Y,Z\rangle p) = \langle \nabla_X Y,Z\rangle p + \langle Y,\nabla_X Z\rangle p + \langle Y,Z\rangle p$ and so

$$\begin{aligned}-R(X,Y)Z &= \langle \nabla_X Y,Z\rangle p - \langle \nabla_Y X,Z\rangle p - \langle [X,Y],Z\rangle p + \langle Y,Z\rangle p - \langle X,Z\rangle p \\ &= \langle Y,Z\rangle X - \langle X,Z\rangle Y.\end{aligned}$$

The sectional curvature is $K = +1$. The Ricci tensor is

$$\mathrm{Ric}(X,Y) = (n-1)\langle X,Y\rangle$$

and the scalar curvature is

$$\mathrm{scal} = n(n-1).$$

Let us find the geodesics. Let γ a geodesics. Then $\nabla_{\dot\gamma}\dot\gamma = 0$. But

$$\begin{aligned}\nabla_{\dot\gamma}\dot\gamma &= (\partial_\gamma\dot\gamma) \\ &= \ddot\gamma^{\text{tangent}} \\ &= \ddot\gamma - \ddot\gamma^{\text{normal}} \\ &= \ddot\gamma - \langle\ddot\gamma,\gamma\rangle\gamma.\end{aligned}$$

After calculus, we find that the geodesics are great circles.

Exemple. The hyperbolic space is $\mathbf{H}^m := \mathbf{R}_+^* \times \mathbf{R}^{m-1}$. Its tangent spaces are $\mathrm{T}_p\mathbf{H}^m \simeq \mathbf{R}^m$. We equip this manifold with the metric

$$g(X,Y) = \frac{\langle X,Y\rangle}{x_1^2}.$$

It is a riemannian manifold with sectional curvature equal to -1.

We can choose others models of the hyperbolic space such as

$$\mathbf{H}^m = \{(x_0,\dots,x_m)\in\mathbf{R}^{m+1} \mid x_0>0,\ -x_0^2+x_1^2+\cdots+x_m^2 = -1\}.$$

Equipped with the induce metric, it is a riemannian manifold. An other model is the Poincaré model

$$\mathbf{D}^m := \{x\in\mathbf{R}^m \mid |x|<1\}$$

with the metric

$$g(X,Y) = \frac{4}{(1-|x|^2)^2}\langle X,Y\rangle.$$

The sectional curvature is also equal to -1.

Exemple. The curvature of $\mathbf{R}_1^m$ is zero, its geodesics are straight lines.

Exemple. We set the pseudo-sphere $\mathbf{S}_\nu^{n-1}\subset\mathbf{R}_\nu^n$. The tangent space is

$$\mathrm{T}_p\mathbf{S}_\nu^{n-1} = \{X\in\mathbf{R}^n \mid \langle p,X\rangle_{\mathbf{R}_\nu^n} = 0\}.$$

The pseudo-spere equipped with the metric $\langle\cdot,\cdot\rangle_{\mathbf{R}_\nu^n}$ is a riemannian manifold of signature $(\nu, n-1-\nu)$. It is diffeomorphic to $\mathbf{R}^\nu\times\mathbf{S}^{n-1-\nu}$ and it sectional curvature is $+1$. The geodesics are branches of hyperboloids, straight line or periodic curves on ellipsoids : we can prove this by considering different cases (the vectors to join are time, space or light like). More over, a curve γ is a geodesic if and only iff the curves $\ddot\gamma$ and γ are parallel.

Chapitre 5

Calculus of variations

Let M be a semi-Riemannian manifold. Let $\gamma\colon I \longrightarrow M$ be a curve. We recall that the set A_γ is the set of maps $Y\colon I \longrightarrow \mathrm{T}M$ along the curve γ, that is such that

$$\forall t \in I, \qquad Y(t) \in \mathrm{T}_{\gamma(t)}M.$$

A such map Y can be write

$$Y(t) = \sum_j \alpha_j(t)(X_j \circ \gamma)(t)$$

on a chart (U, x^i) with $X_j := \partial/\partial x^j$. The derivation for A_γ is

$$\frac{D}{\mathrm{d}t}Y(t) = \sum_k \Big(\dot\alpha_k(t) + \sum_{i,j} \Gamma^k_{ij}(\gamma(t))\dot\gamma_i(t)\alpha_j(t)\Big) X_k(\gamma(t)).$$

Facts.

1. For all $X_0 \in \mathrm{T}_{\gamma(0)}M$, there exists $Y \in A_\gamma$ such that $Y(0) = X_0$ and $\frac{D}{\mathrm{d}t}Y = 0$.
2. For all $X_0 \in \mathrm{T}_aM$, there exists $t_0 > 0$ and $\gamma\colon [0, t_0[\longrightarrow M$ such that $\gamma(a) = X_0$ and $\frac{D}{\mathrm{d}t}\gamma = 0$. Such a γ is called a *geodesic*.
3. We also write $\frac{D}{\mathrm{d}t}Y = Y' = \dot Y = \nabla_{\dot\gamma}Y$.
4. If γ is a geodesic, then
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\dot\gamma, \dot\gamma\rangle = \langle\ddot\gamma, \dot\gamma\rangle = 0.$$

Définition 5.1. Let M be a semi-Riemannian manifold. A *variation* of a function $\alpha\colon [a, b] \longrightarrow M$ of class $\mathscr{C}^\infty$ is a map $x\colon [a, b] \times]-\delta, \delta[\longrightarrow M$ of class $\mathscr{C}^\infty$ with $\delta > 0$ such that $x(u, 0) = \alpha(u)$. The *variation vector field* is the vector field V such that

$$V(u) := \frac{\partial x}{\partial v}(u, 0).$$

The *length* of α is

$$L(\alpha) := \int_a^b |\alpha'(s)| \,\mathrm{d}s$$

where $|\cdot| = \sqrt{|\langle\cdot,\cdot\rangle|}$. The *length* of V is

$$L(v) = L_x(v) = \int_a^b \left|\frac{\partial x}{\partial u}(s, v)\right| \mathrm{d}s.$$

We consider curves such that $|\gamma'(t)| > 0$, called *regular curves of space-like*. We denote ε the sign of $\langle\alpha', \alpha'\rangle$.

Lemme 5.2. If x is a variation of α with $|\alpha'| > 0$, then

$$L_x'(0) = \varepsilon \int_a^b \langle \frac{\alpha'(u)}{|\alpha'(u)|}, V'(u)\rangle \,\mathrm{d}u.$$

Démonstration. With $x_u = \frac{\partial x}{\partial u}$, we have

$$L(u) \coloneqq \int_a^b |x_u(u,v)|\,\mathrm{d}u.$$

We have $\alpha' = x_u(u,0)'$. So for δ small enough, we have $|x_u(u,v)| > 0$ for $u \in \left]-\delta,\delta\right[$. So

$$L'(0) = \int_a^b \frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0} |x_u|\,\mathrm{d}t.$$

But we get

$$\frac{\mathrm{d}}{\mathrm{d}u}|x_u| = \frac{1}{2}(\varepsilon\langle x_u, x_u\rangle)^{-1/2} 2\varepsilon\langle x_u, x_{uv}\rangle = \frac{\varepsilon\langle x_u, x_{uv}\rangle}{\langle x_u, x_u\rangle}.$$

Take $u = 0$, we get $x_u(u,0) = \alpha'(0)$ and $x_v(u,0) = V(u)$ and $x_{uv}(u,0) = V'(u)$. ◇

Proposition 5.3 *(first variation).* Let $\alpha\colon [a,b] \longrightarrow M$ be a continuous and smooth curve piece-wise of constant speed $c > 0$ and of sign ε. Let x be a variation of α. Then

$$L'(0) = -\frac{\varepsilon}{c}\int_a^b \langle\alpha'', V\rangle\,\mathrm{d}u - \frac{\varepsilon}{c}\sum_{i=1}^n \langle\Delta\alpha'(U_i), V(U_i)\rangle + \frac{\varepsilon}{c}\langle\alpha', V\rangle|_a^b$$

with $U_1 < \cdots < U_k$ are points where α is not $\mathscr{C}^\infty$ and

$$\Delta\alpha'(U_i) = \alpha'(U_i^+) - \alpha'(U_i^-) \in \mathrm{T}_{\alpha(U_i)}M.$$

Démonstration. We have

$$\langle\frac{\alpha'}{|\alpha'|}, V\rangle = \frac{1}{c}\langle\alpha', V'\rangle.$$

On $\left]U_i, U_{i+1}\right[$, we have

$$\langle\alpha', V'\rangle = \frac{\mathrm{d}}{\mathrm{d}u}\langle\alpha', V\rangle - \langle\alpha'', V\rangle.$$

So

$$\int_{U_i}^{U_{i+1}} \langle\alpha', V'\rangle\,\mathrm{d}u = \langle\alpha', V\rangle_{U_i}^{U_{i+1}} - \int_{U_i}^{U_{i+1}} \langle\alpha'', V\rangle\,\mathrm{d}u.$$

We sum up to obtain the desired formula. ◇

Corollaire 5.4. A piece-wise smooth curve α with constant speed $c > 0$ is a geodesic if and only if the first variation of L is zero for any variation with fixed ends.

Remarque. Fixed ends imply that V is zero at a and b and

$$\frac{\varepsilon}{2}\langle\alpha', V\rangle|_a^b = 0.$$

Démonstration. Suppose that α is a geodesic, that is $\alpha'' = 0$. Then α is smooth, so $\Delta\alpha'(U_i) = 0$. In particular, we get $V(a) = V(b) = 0$ and so $L'(0) = 0$.

Suppose that $L'(0) = 0$. First we show that α is a geodesic on $\left]U_i, U_{i+1}\right[$, that is $\alpha''(t) = 0$ for $t \in \left]U_i, U_{i+1}\right[$. Let y be in $\mathrm{T}_{\alpha(t)}M$ and f a smooth function defined on $[a,b]$ with $\operatorname{supp} f \subset [t-\delta, t+\delta] \subset \left]U_i, U_{i+1}\right[$ and $f \in [0,1]$ and $f = 1$ on $\left]t-\delta/2, t+\delta/2\right[$. Let Y be the vector field obtained by parallel transport of y along α, that is $\frac{D}{\mathrm{d}t}Y = 0$ and $Y(t) = 0$. Let $V \coloneqq fY$. Then $V(a) = V(b) = 0$. Let exp be the exponential map, that is the map

$$\exp_p\colon D \subset \mathrm{T}_pM \longrightarrow M$$

with $p \in M$ where

$$\exp_p(v) = B(1)$$

where B is the geodesic starting at p with initial speed v and where

$$D = \{v \in \mathrm{T}_pM \mid B(1) \text{ exists}\}.$$

Let $x(u,v) = \exp_{\alpha(u)}(vV(u))$. Then $x(u,v)$ is a variation of α with fixed ends. So $L'(0) = 0$ and then

$$0 = \int_a^b \langle\alpha'', v\rangle\,\mathrm{d}u$$

$$= \int_{t-\delta}^{t+\delta} \langle \alpha'', fY \rangle.$$

This implies that

$$\forall y \in \mathrm{T}_{\gamma(t)}M, \qquad \langle \alpha''(t), y \rangle = 0$$

and so $\alpha''(t) = 0$ on each $]U_i, U_{i+1}[$.

If $y \in \mathrm{T}_{\alpha(U_i)}M$, let f have its support in $]U_{i-1}, U_{i+1}[$ with $f = 1$ around U_i. So

$$0 = L'(0) = -\frac{\varepsilon}{c}\langle \Delta\alpha'(U_i), y \rangle, \qquad \forall y$$

and so

$$\Delta\alpha'(U_i) = 0. \qquad \diamond$$

We will compute $L''(0)$ if $L'(0) = 0$. Any vector field Y along α decomposes as $Y = Y^T + Y^\perp$ where $Y^T = \varepsilon\langle Y, \alpha' \rangle \alpha' =: f\alpha'$ and $Y^\perp$ is orthogonal to α'. If α is a geodesic, then

$$Y' = f'\alpha' + (Y^\perp)'.$$

Moreover, we have $(Y')^\perp = (Y^\perp)'$.

Théorème 5.5 *(second variation).* Let γ be a geodesic of constant speed $c > 0$ and of sign ε. If x is a variation of γ, then

$$L''(0) = \frac{\varepsilon}{c}\int_a^b \langle V'^\perp, V'^\perp \rangle - \langle R(V, \gamma')V, \gamma' \rangle \,\mathrm{d}u + \frac{\varepsilon}{c}\langle \gamma', A \rangle|_a^b$$

where $V(u) = x_v(u, 0)$ and $A(u) = x_{vv}(u, 0)$.

Let $\Omega(p, q)$ be the space of smooth piece-wise curves from $[a, b]$ to M starting at p and ending at q. The *tangent space* to $\Omega(p, q)$ at α is the set $\mathrm{T}_\alpha\Omega(p, q)$ of vector fields V along α with $V(a) = V(b) = 0$. The index of $\sigma \in \Omega(p, q)$ is the bilinear symmetric form

$$I_\sigma \colon \mathrm{T}_\sigma\Omega \longrightarrow \mathrm{T}_\sigma\Omega$$

such that $I_\sigma(V, V) = L_x(\sigma)$ where x is a variation with fixed ends and variation vector V, that is

$$I_\sigma(V, W) = \frac{\varepsilon}{c}\int_a^b \langle V'^\perp, V'^\perp \rangle - \langle R(V, \sigma')W, \sigma' \rangle \,\mathrm{d}u.$$

We have $I_\sigma(V, W) = I_\sigma(V^\perp, W^\perp)$.

Lemme 5.6. Let σ be a non-null geodesic with sign ε. Let M be a semi-Riemannian manifold with index ν. Then

1. if I_σ is semi-definite positive, then $\nu = 0$ or n ;
2. if I_σ is semi-definite negative, then $\nu = 1$ or $n - 1$.

Définition 5.7. Let γ be a geodesic. A vector field Y along γ is called *Jacobi field* if

$$Y'' = R(Y, \gamma')\gamma'.$$

If x is a variation of γ such that

$$\forall v, \qquad u \longmapsto x(u, v) \text{ is a geodesic,}$$

then the variation vector $u \longmapsto \frac{\partial x}{\partial v}(u, 0)$ is a Jacobi field.

For all $v, w \in \mathrm{T}_pM$, there exists an unique Jacobi field Y along γ such that $Y(0) = v$ and $Y'(0) = w$.

Définition 5.8. Two points $\sigma(a)$ and $\sigma(b)$ with $a \neq b$ on a geodesic σ are *conjugate* if there exists a nontrivial Jacobi field Y such that $J(a) = J(b) = 0$.

Then $\sigma(a)$ and $\sigma(b)$ are conjugate if and only if there exists a variation x of σ such that the map $u \longmapsto x(u, v)$ is a geodesic, for all v, started from $\sigma(a)$ such that $\frac{\partial x}{\partial u}(b, 0) = 0$. This is equivalent to the fact that the exponential map $\exp_p \colon \mathrm{T}_pM \longrightarrow M$ is singular at $b\sigma'(0)$, that is there is a tangent vector x to p at $b\sigma'(0)$ such that $d(\exp_p)_{b\sigma'(0)}(x) = 0$.

Lemme 5.9. Let σ be a geodesic such that $\sigma^{\perp}(s) \in \mathrm{T}_{\sigma(s)}M$ is space-like. If $\langle R(v,\sigma')v,\sigma'\rangle \leqslant 0$ for all $v \perp \sigma'$, then there is no conjugate points along σ.